To bee or not to be…

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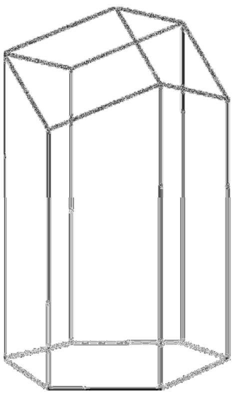
http://www.gschmidt.se/Skapelsefragan/Fran_skapelsens_smorgasbord/En_bisak/En_bisak.html

http://upload.wikimedia.org/wikipedia/commons/3/39/Imker.jpg

The picture above shows a honeycomb constructed by honey bees *(Apis mellifera)*. The wax cake consists of lots of small compartments – or cells – where the bees store nourishment like honey and pollen in some of the cells, and larvae in others. On the picture below you can see a sketch of the geometry of a cell. The structure is with a mathematical nomenclature called a six sided prism, which ends with a three sided pyramid consisting of three rhombic areas. The structure as a whole generally is referred to as *Maraldi's prisma*.



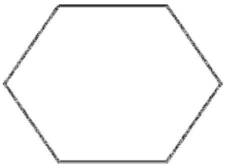
The thickness of the walls is 0,073 mm for the workers' cells and 0,094 mm for the drones', with a deviation of less than 4 per cent, while the bottom areas being somewhat thicker. The depth of the cells are 12-13 mm, depending on if it´s workers' or drones' cells.

It's the lot of man to maze at the miracles of creation. The architectural skill of the bees is one of them. Welcome to join me on a little mathematical journey of investigating the geometry of the bee cells to see what it may tell us about the origin of our honey bees.

The entrances

As can be seen in the picture above, the entrances have the form of regular hexagons. A natural question arises: Why hexagons? Why not trigons (triangles), tetragons (4 sides), pentagons (5 sides)- or nonadecagons (19 sides), or perhaps completely round cell circumferences?

There are only three geometrical structures that allow direct contact with all neighboring cells, and that is tri-, tetra- and hexagons. All other alternatives will lead to spaces between the cells that would gather bacteria and other parasites. It's easy to prove that the hexagon has the smallest



circumference of the three under the condition of constant surface area. So the bees seem to have “chosen” the optimal design of the entrance with respect to material economy.

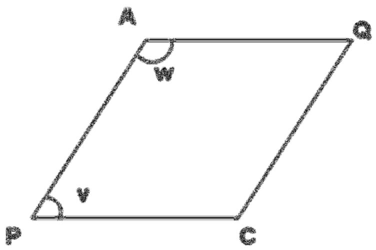
One could add to the discussion that if the bees had built their cells with tetragonal cell entrances - and thus 4-sided prisms - the result would have been an instable geometric structure. The honeycomb would fold when exposed to a bump. If so, we would expect the bees to be much flatter than they are today ;).

So we come to the conclusion that the honey bees have found the same solution that a modern construction engineer or architect would – the hexagon is an optimal choice!

2. The bottom of the cells

The reason that the bottoms of the cells attain the form of a pyramid is that the bees build their honeycombs with cells on both sides, and to make it possible for the bottoms to meet one another wall-to-wall each bottom must meet three other cells from the opposite side of the comb. Thus three bottom areas of each cell.

The bottom area consists of three rhombic areas, which means that they have the form of “tilted squares”. In a square all four angles are right, i. e. 90 degrees. Instead, the rhombuses of the cell bottom have different angles (w in the figure) that are obtuse – 110 degrees, and two (v) that are acute – 70 degrees.



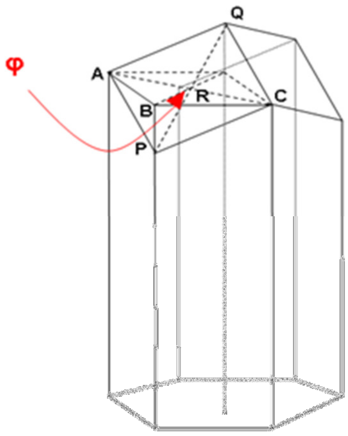
Of course it’s possible to imagine many different pairs of angles in the corners of these rhombuses, but once again a suspicion appears – Might there be a rational explanation to why honey bees all over the world happen to construct their cells with 70 and 110 degrees in the corners of their rhombuses? Is it possible that the reason is material economy even this time?

We could check it up by choosing any arbitrary angle *v* and then calculate the surface area of the cell. Then choose another angle and repeat the calculation and continue doing so in an intelligent manner successively coming closer and closer to the angles which entail the optimal proportions of the cell with respect to material expenditure. But instead of this method of trial and error (or letting a computer iterate towards this goal) we pull our college math book out of the shelf, browse through the pages until we find our favorite chapter – derivatives – and get off!

Let’s start by calculating the surface area of the cell walls:

Suppose that each of the sides in the hexagon is a length units (l.u.). This means that in the figure below AB = BC = a

The angle ABC = 120 ° which means that the angle RBC = 60°, which in turn gives that



- the distance RB = $a \cdot \cos 60^\circ = \frac{a}{2}$
- the distance AC = $2 \cdot a \cdot \sin 60^\circ = \sqrt{3} \cdot a$
- the distance RC = $\frac{(AC)}{2} = \frac{\sqrt{3}}{2} \cdot a$
- the distance BP = $\frac{a}{2} \cdot \tan \varphi$
- the distances PR = RQ = $\frac{a}{2 \cdot \cos \varphi}$

Now, if the length of the cell is h l.u. each wall in the cell will have the area $A_1(\varphi)$ a.u. where $A_1(\varphi) = a \cdot h - \frac{(BC) \cdot (BP)}{2} = a \cdot h - \frac{a \cdot \frac{a}{2} \cdot \tan \varphi}{2} = a \cdot h - \frac{a^2 \cdot \tan \varphi}{4}$ a.u.

The area of a rhombus can be calculated by multiplying the lengths of the two diagonals divided by two, i. e. the area of each rhombus in the bottom wall is $A_2(\varphi)$, where

$$A_2(\varphi) = \frac{(AC) \cdot (PQ)}{2} = \frac{(AC) \cdot 2(PR)}{2} = \frac{a \cdot \sqrt{3} \cdot 2 \cdot (\frac{a}{2 \cdot \cos \varphi})}{2} = \frac{\sqrt{3} \cdot a^2}{2 \cdot \cos \varphi} \text{ a.u.}$$

The total limit area of the cell A consists of the sum of the six side walls and the three rhombuses in the bottom, i.e.:

$$A(\varphi) = 6 \cdot A_1(\varphi) + 3 \cdot A_2(\varphi) = 6 \cdot (a \cdot h - \frac{a^2 \cdot \tan \varphi}{4}) + 3 \cdot (\frac{\sqrt{3} \cdot a^2}{2 \cdot \cos \varphi}) = 3 \cdot \frac{a^2}{2} \cdot (\frac{\sqrt{3}}{\cos \varphi} - \tan \varphi) + 6 \cdot a \cdot h$$

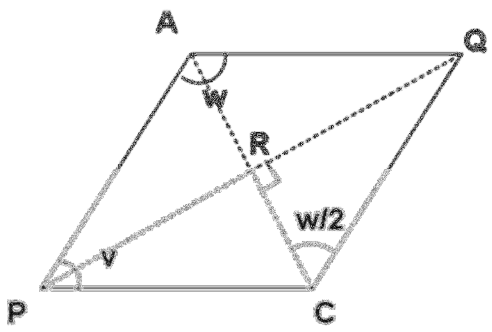
a.u. $(0^\circ < \varphi < 90^\circ)$

The area function $A(\varphi)$ has a minimum when the expression $(\frac{\sqrt{3}}{\cos \varphi} - \tan \varphi)$ has a minimum. Let us from now on call this $f(\varphi)$.

Now let us formulate an expression for the derivative $f'(\varphi)$ and find out which angles φ make the derivative become zero (a classical optimization method):

$$f'(\varphi) = \sqrt{3} \cdot \frac{\sin \varphi}{\cos^2 \varphi} - \frac{1}{\cos^2 \varphi} = \frac{\sqrt{3} \cdot \sin \varphi - 1}{\cos^2 \varphi}$$

It's obvious that $f'(\varphi) = 0$ when the expression $(\sqrt{3} \cdot \sin \varphi - 1)$ equals 0 i.e. when



$\sin\varphi = \frac{1}{\sqrt{3}}$ which renders $\varphi \approx 35{,}3^\circ$

From the figure it becomes apparent that:

$$\tan\frac{w}{2} = \frac{(RQ)}{(RC)} = \frac{\frac{a}{2\cdot\cos\varphi}}{\frac{\sqrt{3}}{2}\cdot a} = \frac{1}{\sqrt{3}\cdot\cos\varphi}$$

Consequently the bee cell's total surface area has a minimum when $\varphi \approx 35{,}3^\circ$, which apply when $\tan\frac{w}{2} = \frac{1}{\sqrt{3}\cdot\cos 35{,}3^\circ} \approx 0{,}707$, which gives $w \approx 70{,}5^\circ$, which in turn gives $v \approx 109{,}5^\circ$

Rounding off to whole integers gives us $v \approx 70^\circ$ and $w \approx 110^\circ$**,** which *"happen"* to be exactly the angles that honey bees use when they construct their cells.

When did you watch a honey bee doing differential calculus lately?

As a matter of fact bees don't sit down deriving when they are about to build their cells. Like on an assembly line hundreds of bees arrive one after another, all delivering a small portion of wax to the growing cell. When the bee is going to decide if the wall thickness is appropriate within a thousandth of a millimeter it does the following:

The bee lowers its head and presses its jaws against the cell wall and causes a small dent in it. As the bee removes its jaws the dent "pops back" with a sound that depends on the thickness of the wall. With the help of its antennae the bee measures the frequency of the sound, and if it's regarded as o.k. the bee leaves (!) and turns to the next cell. Researchers have tested this by cutting off the tips of the antennae of the poor creatures. As a result the walls of the cells (though still hexagonal) will vary in thickness.

Now, what conclusion can be drawn from this?

Well, that depends on one's perspective of the origin issue. The Darwinist shrugs his shoulders as usual and states without doubt that mutations and natural selection have the ability to chisel out this kind of optimal structure in the living world. They reason that the descendants of a bee population which by pure chance happened to be equipped with a genetic program that instinctively urged their members to build their cells with the completely disastrous angels 111 and 69 degrees respectively, would eventually be competed out and eliminated by a hypothetical neighbor population which by the same chance happened to use the mathematically correct angles. Let us for purely pedagogical reasons call these poorly adapted bees *the clumsies*, and the lucky ones *the professionals*.

The Darwinists' argumentation is relevant only as far as one takes just one single factor into account - namely the angles in the corners of the rhombs – leaving all other factors out of the discussion. That scenario is next to absurd for several reasons:

First of all it's well known that *the professionals* must be rather **lucky** to manage to transfer their genes onto the next generation. It presupposes that Winnie the Poe, preparing for his hibernation searching for carbohydrates, chooses to rob *the clumsies'* hive instead of *the professionals'*. Of course Winnie won't sit down measuring the angles in the corners of rhombs in the honeycombs before choosing which hive to drain for honey. He neither could nor would be interested in doing so. Apparently there is a considerable factor of chance involved.

Secondly there are innumerable other factors involved, whose effects on the chances of survival of the bee population greatly exceed the saved labor resulting from the effect of that single degree. Let's say, for example, that *the clumsies* happened to build their hive 10 feet closer to a population of nectar-rich flowers than *the professionals*. The reduced flight distance of 20 feet each collection round would carry a considerably larger energy saving for the population than the "defect" angle would cost. Or say that *the clumsies* happened to build their hive on a location with some tenth of a centigrade higher mean temperature than *the professionals*. That energy saving would surely also exceed that caused by the defective angle. And examples like these could be multiplied. The surrounding "noise" from innumerable environmental sources simply "drown" the "signal" that some angle degree of improved cell geometry would entail.

The bees' cell is only one of innumerable examples of the "wonders" of nature. With the difference that it is a relatively accessible one from a mathematical point of view.

The conclusion from this argumentation is that **natural selection lacks the power to chisel out nature's structures with the resolution needed to accomplish the "miracles of nature".**

The honey bees bear (!) the witness of their Creator!

Epilogue

After one of my lectures when I had presented this example, a defender of evolution commented that the six sided prisms are exactly the same form as soap bubbles or soft balls take on when they are compressed in some kind of a container. That argument would definitely have been worth considering *if* the bees' honeycombs were manufactured during a process of air bubbles in liquid wax being compressed all at once.

However – this is not the case!

The bees successively and methodically construct their honeycombs in the manner I have portrayed above during a longer period of time without any sign that the construction ever prevails in some half-liquid state. Which in turn makes this argument fall flat.

The fact that the mathematically optimal angels *also* occur when a physical system spontaneously attains its lowest energy state during a bubble compression is rather to be considered as an independent confirmation – beside the mathematical optimization method above – that the bees' programming is astounding.

There is, however, another objection mentioned in the literature, one that I am willing to agree upon. It is that the mathematical calculations presented in this article refer to an idealized bee cell. Especially in the periphery of the honeycomb there are cells that are not as symmetrical as those in the more central parts. But that objection does not affect the argumentation as a whole. The angles in an average bee cell are sufficiently close to the idealized one for the calculations to be relevant.

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